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On the Quantum Moduli Space of Vacua of $N = 2$ Supersymmetric $SU(N_c)$ Gauge Theories

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Abstract

We construct families of hyper-elliptic curves which describe the quantum moduli spaces of vacua of $N = 2$ supersymmetric $SU(N_c)$ gauge theories coupled to N_f flavors of quarks in the fundamental representation. The quantum moduli spaces for $N_f < N_c$ are determined completely by imposing R -symmetry, instanton corrections and the proper classical singularity structure. These curves are verified by residue and weak coupling monodromy calculations. The quantum moduli spaces for $N_f \geq N_c$ theories are parameterized and their general structure is worked out using residue calculations. Global symmetry considerations suggest a complete description of them. The results are supported by weak coupling monodromy calculations. The exact metrics

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on the quantum moduli spaces as well as the exact spectrum of stable massive states are derived. We find an example of a novel symmetry of a quantum moduli space: Invariance under the exchange of a moduli parameter and the bare mass. We apply our method for the construction of the quantum moduli spaces of vacua of $N = 1$ supersymmetric theories in the coulomb phase.

1 Introduction

There has been much progress recently in the study of non-perturbative properties of $N = 1$ and $N = 2$ four dimensional supersymmetric field theories. Of particular importance to the analysis is the concept of holomorphy [1]. In the $N = 1$ theories, the super-potential and the coefficients of the gauge kinetic terms are holomorphic and are therefore much constrained. In the $N = 2$ theories, the Kähler potential is also constrained by holomorphy, thus making the holomorphy even more powerful.

A basic ingredient in the analysis is the moduli space of vacua corresponding to a continuous degeneracy of inequivalent ground states. Classically, the super-potential has flat directions along which the squarks get vacuum expectation values and thus break the gauge symmetry. The singularities of the classical moduli space are of two types: They either correspond to enhanced gauge symmetry where the gauge group is not broken to its maximal torus, or correspond to vacua where some of the matter fields become massless.

The quantum moduli space is in general different from the classical moduli space. In [2] the quantum moduli space of $N = 2$ supersymmetric pure $SU(2)$ gauge theory was determined as a certain one parameter family of elliptic curves. The exact metric on the quantum moduli space as well as the exact spectrum of BPS particles were found via the periods of the curves. The two singularities of the quantum moduli space were associated with massless monopole and dyon. The generalization of [2] to include massless and massive matter hyper-multiplets was carried out in [3]. The quantum moduli space of $N = 2$ supersymmetric pure $SU(N_c)$ gauge theories was constructed as an $N_c - 1$ parameter family of hyper-elliptic curves in [4, 5], and the physics of the model was studied in [6].

Our aim in this paper is to determine the quantum moduli spaces of vacua of the coulomb phase of $N = 2$ supersymmetric $SU(N_c)$ gauge theories with N_f matter hyper-multiplets in the fundamental representation, as well as the exact metric on the quantum moduli space and the exact spectrum of massive stable particles. The quantum moduli spaces will be constructed as families of hyper-elliptic curves satisfying a set of physical constraints: R -symmetry, singularity structure and global symmetries, appropriate inclusion of instanton correction, proper classical and scaling (integration of massive quarks) limits, compatibility of residue calculations with the BPS formula and correct weak coupling monodromies.

The paper is organized as follows. In section two we review general aspects of $N = 2$ gauge theories and the moduli spaces of vacua, introduce the physical quantities associated

with them, discuss their singularity and monodromy structures, and present principles for their construction. In section three we construct the quantum moduli space for $N_f < N_c$. We show that it is determined completely by imposing R -symmetry, instanton corrections and the proper classical singularity structure. We get compatibility of the residue calculations with the BPS formula which provides a consistency check of the result. Section four is devoted to the $N_c = N_f$ theories. Applying the physical constraints including residue calculations leaves us with one undetermined constant parameter. In the $SU(2)$ case it is determined by imposing an appropriate Z_2 symmetry in the moduli space. For $N_c > 2$ we determine the parameter by requiring compatibility, upon integration of a massive quark, with the $N_f > N_c$ theories which are worked out in the subsequent section.

In section five the quantum moduli spaces for $N_f > N_c$ theories are parameterized and their general structure is worked out using residue calculations. This still leaves us with undetermined constant coefficients. Global symmetry considerations suggest a complete determination of them, which we conjecture but do not prove.

The results of the previous sections are supported by weak coupling monodromy calculations in section six. In section seven we discuss the $N_c = 2N_f$ theories. When the bare masses are zero one expects scale invariant theories with periods satisfying the classical relations. The quantum moduli spaces in the massless as well as the massive theories are parameterized and their general structure is worked out using residue calculations in a similar manner to the $N_f > N_c$ case. Again, global symmetry considerations suggest a complete description of the quantum moduli spaces which we do not prove. The examples of quantum moduli spaces for $N_c = 2, 3$ are worked out in detail in each section.

In general our results provide the exact metrics on the quantum moduli spaces as well as the exact spectrum of stable massive states. Along the way we find an example of a novel symmetry of a quantum moduli space: Invariance under the exchange of a moduli parameter and the bare mass. Section eight is devoted to discussion and conclusions.

The method that we use in order to construct the quantum moduli spaces of $N = 2$ theories is rather general. It can be used in order to construct the quantum moduli spaces of a large class of $N = 1$ theories, in the coulomb phase, with different matter content. In the appendix, as an example, we apply our method for constructing the quantum moduli spaces, to $N = 1$ supersymmetric $SU(N_c)$ gauge theories with a single matter field in the adjoint representation of the gauge group and N_f matter fields in the fundamental representation. The results generalize those of [7] to $N_c > 2$.

2 The moduli space of vacua of $N = 2$ gauge theories

In this section we review general aspects of $N = 2$ gauge theories and the moduli spaces of vacua, introduce the physical quantities associated with them, discuss their singularity and monodromy structures, and present principles for their construction.

2.1 $N = 2$ QCD and the moduli space of vacua

We will consider $N = 2$ supersymmetric $SU(N_c)$ gauge theories with N_c colors and N_f flavors. The field content of the theories consists of $N = 2$ chiral multiplet and N_f hyper-multiplets. The $N = 2$ chiral multiplet contains gauge fields A_μ , two Weyl fermions λ and ψ (alternatively one Dirac fermion) and a complex scalar ϕ , all in the adjoint representation of the gauge group $SU(N_c)$. The $N = 2$ hyper-multiplets that we will consider contain two Weyl fermions ψ_q and $\psi_{\tilde{q}}^\dagger$ and two complex bosons q and \tilde{q}^\dagger , all in the fundamental representation of $SU(N_c)$.

In terms of $N = 1$ super-fields the $N = 2$ chiral multiplet consists of a vector multiplet W_α and a chiral multiplet Φ , while the $N = 2$ hyper-multiplets consist of two chiral super-fields Q_a^i and \tilde{Q}_{ia} , where $i = 1, \dots, N_f$ is the flavor index and $a = 1, \dots, N_c$ is the color index. The super-potential in the $N = 1$ language reads

$$W = \sqrt{2}\tilde{Q}_i\Phi Q^i + \sum_i m_i\tilde{Q}_iQ^i, \quad (2.1)$$

with m_i being the bare masses and color indices are suppressed. The first term in (2.1) is related by $N = 2$ super-symmetry to the gauge couplings and the second term corresponds to $N = 2$ invariant mass terms. When the quarks are massless the global symmetry of the classical theory is a certain subgroup of $SU(N_f) \times SU(N_f) \times SU(2)_R \times U(1)_R$.

The theory has an $N_c - 1$ complex dimensional moduli space of vacua, which are parameterized by the gauge invariant order parameters

$$u_k = Tr\langle\phi^k\rangle, \quad k = 2, \dots, N_c, \quad (2.2)$$

ϕ being the scalar field of the $N = 2$ chiral multiplet. The moduli space of vacua corresponds to ϕ satisfying a D-flatness condition $[\phi, \phi^\dagger] = 0$. Thus, up to gauge transformation we can take

$$\langle\phi\rangle = \sum_{i=1}^{N_c} a_i H_i = diag[a_1, \dots, a_{N_c}], \quad (2.3)$$

with H_i being the generators of the Cartan sub-algebra of $U(N_c)$ and

$$\sum_{i=1}^{N_c} a_i = 0. \quad (2.4)$$

At weak coupling we have

$$u_k = \sum_{i=1}^{N_c} a_i^k . \quad (2.5)$$

Alternative gauge invariant order parameters which will be useful for $N_c > 3$ are defined at the classical level as the symmetric polynomials s_k in a_i

$$s_k = (-)^k \sum_{i_1 < \dots < i_k} a_{i_1} \cdots a_{i_k}, \quad k = 2, \dots, N_c , \quad (2.6)$$

Using (2.5) and (2.6) the sets s_k and u_k are related by Newton's formula [5]

$$ks_k + \sum_{i=1}^k s_{k-i} u_i = 0, \quad k = \{1, 2, \dots\} , \quad (2.7)$$

where $s_0 = 1, s_1 = u_1 = 0$. This relation serves as a definition of s_k at the quantum level.

At a generic point the expectation values of ϕ break the gauge symmetry to $U(1)^{N_c-1}$ and a low-energy effective Lagrangian can be written in terms of multiplets (A_i, W_i) , $i = 1, \dots, N_c$, where $\sum_i A_i = 0$.

The $N = 2$ effective Lagrangian takes the form

$$\mathcal{L}_{eff} = Im \frac{1}{4\pi} \left[\int d^4\theta \partial_i \mathcal{F}(A) \bar{A}^i + \frac{1}{2} \int d^2\theta \partial_i \partial_j \mathcal{F}(A) W^i W^j \right] , \quad (2.8)$$

where \mathcal{F} is a holomorphic pre-potential.

Classically,

$$\mathcal{F}_{cl}(A) = \frac{\tau}{2} \sum_{i=1}^{N_c} \left(A_i - \frac{1}{N_c} \sum_{j=1}^{N_c} A_j \right)^2 , \quad (2.9)$$

where $\tau = N_c \left(\frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \right)$. The one loop correction for this pre-potential \mathcal{F} is given by

$$\mathcal{F}_1 = i \frac{2N_c - N_f}{8\pi} \sum_{i < j} (A_i - A_j)^2 \log \frac{(A_i - A_j)^2}{\Lambda^2} , \quad (2.10)$$

and $N = 2$ super-symmetry implies that there are no perturbative corrections beyond one loop. There are, however, non-perturbative instanton corrections.

The classical moduli space is described by the family of genus $g = N_c - 1$ hyper-elliptic curves [4, 5] *

$$y^2 = \mathcal{C}_{N_c}^2(x) = \prod_{i=1}^{N_c} (x - a_i)^2 , \quad (2.11)$$

*Note that the curve (2.11) differs from $y^2 = (x^{N_c} - \sum_{i=2}^{N_c} \frac{u_i}{i} x^{N_c-i})^2$ for $N_c > 3$.

and in terms of the symmetric functions s_i

$$\mathcal{C}_{N_c}(x) = x^{N_c} + \sum_{i=2}^{N_c} s_i x^{N_c-i} . \quad (2.12)$$

In (2.11) y is a double cover of the x plane branched at N_c points corresponding to the roots a_i . The singularities of the classical curve correspond to cases where the gauge group is spontaneously broken to non-Abelian subgroups of $SU(N_c)$ rather than to $U(1)^{N_c-1}$. Since each root has multiplicity greater than one in (2.11), the classical curve is always singular. Physically, this is related to the fact that the dynamical scale of the theory vanishes and the states with mass proportional to the scale become massless.

We will construct the quantum moduli spaces of vacua also as families of hyper-elliptic curves

$$y^2 = \prod_{i=1}^{2g+2} (x - e_i) , \quad (2.13)$$

where the roots e_i are functions of the quark masses m_i , the dynamically generated scale Λ [†] and the gauge invariant order parameters u_k or s_k , with (2.11) being their classical limit.

2.2 Dyon spectrum and duality

The spectrum of $N = 2$ supersymmetric QCD includes particles in the “small” representation of the $N = 2$ algebra, the so called BPS-saturated states. These are electrically and magnetically charged particles with masses

$$M^2 = 2|Z|^2 , \quad (2.14)$$

where Z is a central extension of the $N = 2$ supersymmetry algebra and is a linear combination of conserved charges [8]. It reads [3]

$$Z = \sum_{i=1}^{N_c} [n_e^i a^i + n_m^i a_D^i] + \sum_{i=1}^{N_f} S_i m_i , \quad (2.15)$$

where n_e, n_m are the electric and magnetic charges satisfying $\sum_{i=1}^{N_c} n_e^i = 0$, $\sum_{i=1}^{N_c} n_m^i = 0$. S_i are the $U(1)$ charges corresponding to additional symmetries that may exist when the global symmetry is explicitly broken by non-zero masses. m_i are the bare masses of the

[†]When $N_f = 2N_c$, the role of the dynamical scale Λ is played by a modular form of the appropriate modular group.

hyper-multiplets and a, a_D correspond to the vacuum expectation values of the scalar component of the chiral super-field A and its dual

$$a_D^i = \frac{\partial \mathcal{F}(a)}{\partial a} . \quad (2.16)$$

Mathematically, a, a_D are sections of a certain bundle over the moduli space of vacua. When all the masses m_i are zero, a, a_D are sections of an $Sp(2g, Z)$ bundle, thus their monodromies upon traversing a closed cycle in the moduli space are elements of $Sp(2g, Z)$. The mass formula (2.14) is $Sp(2g, Z)$ invariant, reflecting a duality property of the theory. When the masses do not vanish, constant shifts are allowed transformations in addition to the above $Sp(2g, Z)$ structure [2, 3]. Thus, the monodromy transformation takes the form

$$\begin{pmatrix} a_D \\ a \end{pmatrix} \rightarrow M \begin{pmatrix} a_D \\ a \end{pmatrix} + c, \quad M \in Sp(2g, Z) , \quad (2.17)$$

where c is independent of the moduli parameters and depends on the masses of the quarks.

a, a_D can be written as periods of a meromorphic one form λ on the curve describing the space of vacua

$$a_D^i = \oint_{\alpha_i} \lambda , \quad a^i = \oint_{\beta_i} \lambda , \quad (2.18)$$

where α_i, β_j form a basis of homology cycles on the curve. In order to determine λ one assumes that

$$\begin{aligned} \frac{\partial a_D^i}{\partial s_k} &\propto \oint_{\alpha_i} x^{N_c-k} \frac{dx}{y} & k = 2, \dots, N_c \\ \frac{\partial a^i}{\partial s_k} &\propto \oint_{\beta_i} x^{N_c-k} \frac{dx}{y} & k = 2, \dots, N_c , \end{aligned} \quad (2.19)$$

with $x^{N_c-k} \frac{dx}{y}$ $k = 2, \dots, N_c$ being a basis of holomorphic one forms on the curve. The exact proportionality factor in (2.19) is determined by matching a^i to the weak coupling relations (2.5) or (2.6). It is a constant since we want to avoid unwanted zeros or poles. Its value, as will be calculated in section six, is $\frac{1}{2\pi i}$.

Note that (2.18) and (2.19) yield, up to an exact form,

$$\frac{d\lambda}{ds_k} \propto x^{N_c-k} \frac{dx}{y} \quad k = 2, \dots, N_c . \quad (2.20)$$

When the bare masses are zero the residues of λ vanish, thus ensuring that a, a_D are invariant under deformations of the cycle of integration even across poles of λ . For non zero masses the residues take the form

$$2\pi i \text{ res}(\lambda) = \sum_i n_i m_i \quad n_i \in \frac{1}{2}Z , \quad (2.21)$$

thus allowing jumps in a, a_D , as defined in (2.18), when crossing poles of λ compatible with the mass formula (2.15).

The matrix of low energy coupling constants, τ , is given by

$$\tau_{ij}(a) = \frac{\partial a_D^i}{\partial a^j} . \quad (2.22)$$

By virtue of (2.19), τ_{ij} is the period matrix of the curve describing the quantum moduli space. The metric on quantum moduli space reads

$$(ds)^2 = Im \, da_D^i d\bar{a}^i , \quad (2.23)$$

is invariant under the transformation (2.17), and is positive definite.

2.3 The singularity, global symmetry and monodromy structures

As discussed in section 2.1, the $N_c - 1$ dimensional moduli space of vacua \mathcal{M}_{N_c} is parameterized by gauge invariant order parameters such as s_k of equation (2.6). The singular locus of the family of curves (2.13) describing the moduli space is the codimension one variety defined as the vanishing locus of the discriminant *

$$\Delta[s_k] \equiv \prod_{i < j} (e_i - e_j)^2 = 0 . \quad (2.24)$$

Along this variety additional massless states appear in the spectrum and the effective low energy description (2.8) is not valid.

In the semi-classical limit $\Lambda \rightarrow 0$ the discriminant factorizes

$$\Delta[\Lambda \rightarrow 0] = \Lambda^{N_c(2N_c - N_f)} \Delta_{N_c}^2 \Delta_{N_f, N_c} , \quad (2.25)$$

where

$$\Delta_{N_c} = \prod_{i < j}^{N_c} (a_i - a_j)^2 , \quad (2.26)$$

and

$$\Delta_{N_f, N_c} = \prod_{j=1}^{N_f} \left(\sum_{i=0}^{N_c} s_i (-m_j)^{N_c - i} \right) . \quad (2.27)$$

The zero locus of Δ_{N_c} corresponds to singularities where classically the $SU(N_c)$ group is not spontaneously broken to $U(1)^{N_c-1}$ but rather to a larger subgroup (In particular, the

*When the coefficient of the highest order monomial x^n in the polynomial is $a \neq 1$ the discriminant (2.24) is modified by a pre-factor a^{2n-2} .

gauge group is not broken at all at the origin of the moduli space). The zero locus of Δ_{N_f, N_c} defines a complex codimension one variety in the moduli space where a quark becomes massless classically. In order to see this note that the singular locus corresponding to a massless quark, $a_i + m_j = 0$, $i = 1, \dots, N_c$, $j = 1, \dots, N_f$, is a codimension one variety defined by

$$\prod_{i=1}^{N_c} (a_i + m_j) \equiv \sum_{k=0}^{N_c} s_k (-m_j)^{N_c-k} = 0, \quad j = 1, \dots, N_f, \quad (2.28)$$

where (2.28) is derived by reading the classical mass of the quarks from the super-potential (2.1) and using the classical curve (2.11)[†]. The product up to N_f in (2.27) is a consequence of having N_f different flavors. The power of Λ in (2.25) follows from instanton contribution and from the fact that classically N_c roots degenerate. The factorization (2.25) holds quantum mechanically with the symmetric functions being modified by quantum corrections.

While singularities of the curves are associated with the appearance of additional massless states in the spectrum, the order of vanishing of the discriminant at a point in the moduli space corresponding to a singularity of the curves is generically the number of codimension one varieties intersecting at the point, which in turn is the number of massless states at that point. These states belong to a representation of the global symmetry group and the order of vanishing of the discriminant should be related to its dimension. Thus, compatibility between the global symmetry and the singularity structure imposes constraints on the latter.

Note, however, that different types of singularities of the curves are associated with different physics. As discussed in [6], when more than two branch points of the curve coincide the massless states at the corresponding point in the moduli space are not mutually local. The singular points corresponding to $N = 1$ vacua, upon adding a mass term as perturbation, are associated with curves of degree $2n$ with $n - 1$ of pairs of identical branch points.

We shall prove in the sequel that the moduli space of vacua for $N_f < N_c$ is fully determined by R -symmetry, instanton contributions and classical singularity without any need for constraints coming from global symmetry considerations, thus consider the cases $N_f \geq N_c$. Let us present an observation that we have about the relation between the global symmetries and the singularity structure, for which we do not have a proof. This observation will be used by us in order to suggest a complete determination of the curves for $N_f \geq N_c$.

[†]In order not to carry factors of $\sqrt{2}$ we re-scale the mass appearing in (2.1) by $m \rightarrow \sqrt{2}m$.

Consider the theories with N_c massless multiplets and $N_f - N_c$ massive multiplets with the same bare mass m . Denote by \mathcal{L} the variety defined by (2.24). Let l be a complex line in the moduli space defined by $s_i = 0$ for all but one modulus, and consider the intersection of \mathcal{L} and l . $\mathcal{L} \cap l$ exhibits a singularity structure with three singular points of multiplicity $(N_c, N_c, N_f - N_c)$. This is probably related to a global symmetry group $SU(N_c) \times SU(N_c) \times SU(N_f - N_c)$.

The singularity structure should also reflect itself in the monodromy structure. The monodromy group of the family of curves (in the massless case) is the homomorphic image of the fundamental group $\Pi_1(\mathcal{M}_{N_c} - \mathcal{L})$ in $Sp(2N_c - 2, Z)$. Physically, the monodromy matrices (in the strong coupling regime) specify the electric and magnetic quantum numbers of the massless dyon associated with the singularity. These electric and magnetic charges are the left eigenvectors of the monodromy matrix with eigenvalue one.

A non trivial monodromy exists also at infinity in the moduli space corresponding to the semi-classical regime of the theory. This monodromy is not associated with additional massless states but rather with the logarithm in (2.8). This monodromy will be computed in section six for arbitrary N_c and N_f , as a check on the curves.

2.4 Principles for the construction of the moduli spaces

In the following we summarize the principles which will be used by us in the construction of the families of hyper-elliptic curves describing the quantum moduli spaces of vacua:

1. **Symmetry** : The curves are invariant under R charge transformation

$$O \rightarrow \exp \left[\frac{2\pi i R(O)}{4(2N_c - N_f)} \right] O , \quad (2.29)$$

where $R(O)$ is the R charge of O and O refers to the building blocks of the curves, y, x, m_i, Λ, s_k . We have

O	y	x	m_i	Λ	u_k	s_k
$R(O)$	$2N_c$	2	2	2	$2k$	$2k$

2. **Singularity structure** : As discussed in section 2.3, singularities of a curve describing a quantum moduli space are associated with the zero locus of the corresponding discriminant and are physically interpreted as the appearance of additional massless states in the spectrum, where the order of vanishing of the discriminant at a point

corresponding to a singularity of the curve is generically the number of massless states at that point. These states belong to a representation of the global symmetry group and the order of vanishing of the discriminant is its dimension. Thus, global symmetry imposes constraints on the singularity structure.

3. **Instanton corrections** : The one instanton process contribution to the curves for $N_f < 2N_c$ takes the form [9]

$$\Lambda^{2N_c - N_f} . \quad (2.30)$$

4. **Integration of a massive quark**: Sending a quark mass $m \rightarrow \infty$ in $N_f < 2N_c$ theories, and sending the scale $\Lambda_{N_f} \rightarrow 0$ such that *

$$\Lambda_{N_f-1}^{2N_c - N_f + 1} = m \Lambda_{N_f}^{2N_c - N_f} , \quad (2.31)$$

is fixed, we reduce the number of flavors from N_f to $N_f - 1$ and we require compatibility of the curves. When $N_f = 2N_c$ the appropriate matching condition reads

$$\Lambda_{2N_c-1} = 64mq , \quad (2.32)$$

where $q \equiv e^{2\pi i \tau}$ and the constant 64 is a choice of a renormalization scheme.

5. **Classical limit**: The curves should exhibit the singularity structure (2.25)-(2.27) at the classical level $\Lambda \rightarrow 0$.
6. **Residues** : As discussed in section 2.2, the meromorphic one form on the curve λ in (2.18) may have poles but its residues are restricted by (2.21). This, as we shall see, leads to powerful constraints on the structure of the curves.

Our aim is to construct the hyper-elliptic curves describing the quantum moduli spaces of $N = 2$ QCD with N_c colors and N_f flavors in the fundamental representation of the gauge group. For this, the form of the hyper-elliptic curves introduced in [4, 5] is the appropriate framework. The general strategy for constructing the curves will be to first restrict the possible terms to those compatible with R charge symmetry and the form of the instanton corrections. We then impose the proper classical singularity structure and use residue calculations. Finally, we make use of the global symmetry and its implications on the singularity structure. Monodromy calculations will be used as a consistency check.

Since the one loop beta function of the theory is proportional to $2N_c - N_f$ (higher loop corrections to it vanish) we limit ourselves to $N_f < 2N_c$ where the theory is asymptotically free and to $N_f = 2N_c$ where the (massless) theory is scale invariant.

*In [10] this renormalization scheme was identified as the \overline{DR} scheme.

3 The quantum moduli space for $N_f < N_c$

3.1 The general case

For $N_f < N_c$ there exist flat directions along which the $SU(N_c)$ gauge group is generically broken to $U(1)^{N_c-1}$ and the theory is in a coulomb phase: There is a collection of $N_c - 1$ massless photons in the spectrum. In these regions, as we shall argue, R -symmetry, classical singularity and the form of the instanton corrections are sufficient in order to fully determine the quantum moduli spaces. First recall the curve for $N_f = 0$. It reads [4, 5]

$$y^2 = \mathcal{C}_{N_c}^2(x) - \Lambda_0^{2N_c} , \quad (3.1)$$

where $\mathcal{C}_{N_c}(x)$ is given by (2.12).

When N_f flavors in the fundamental representation of the gauge group are present such that $N_f < N_c$ we have the following claim:

Claim: The curve describing the quantum moduli space with $N_f < N_c$ flavors is:

$$y^2 = \mathcal{C}_{N_c}(x)^2 - \Lambda_{N_f}^{2N_c-N_f} \prod_{i=1}^{N_f} (x + m_i) . \quad (3.2)$$

Proof: R -symmetry, the form of instanton corrections together with the classical singularity of the gauge group imply that the most general curve is

$$y^2 = \mathcal{C}_{N_c}(x)^2 - \Lambda_{N_f}^{2N_c-N_f} G(x, m_i) \equiv P , \quad (3.3)$$

where $G(x, m_i)$ is a polynomial of degree N_f in x . A priori, G may also depend on the moduli s_k . In the sequel it will be shown that in fact it is independent of the moduli. The second term in (3.3) is the quantum correction to the classical curve (2.11). Note that there are only one instanton contributions to the curve (3.3) due to the R charge restriction. The polynomial $G(x, m_i)$ is determined by requiring that the discriminant of the polynomial P in (3.3) has the right classical limit (2.25). In order to do this we need to evaluate first the classical limit of the discriminant. Note that the discriminant of a polynomial can be evaluated up to a multiplicative constant by *

$$\Delta[P] = \prod_{x_i \in S} P(x_i) , \quad (3.4)$$

where $x_i \in S$ are the critical points of P , $\partial_x P(x_i) = 0$. Differentiating the polynomial in (3.3) with respect to x and equating to zero we have

$$P' = 2\mathcal{C}_{N_c}(x)\mathcal{C}_{N_c}'(x) - \varepsilon G(x, m_i)' = 0 , \quad (3.5)$$

*The discriminant of a degree n curve is a polynomial in its roots of degree $n(n-1)$ that vanishes when any two roots of the curve coincide. It is easy to see that (3.4) satisfies these.

where we denoted $\varepsilon = \Lambda_{N_f}^{2N_c - N_f}$. The roots of equation (3.5) are $\{r_i = a_i + \varepsilon \text{ corrections}\}$ and $\{s_i = b_i + \varepsilon \text{ corrections}\}$ where a_i and b_i are the roots of $\mathcal{C}_{N_c}(x)$ and $\mathcal{C}_{N_c}(x)'$, respectively. According to (3.4)

$$\Delta[P] = \prod_i P(r_i) \prod_j P(s_j) . \quad (3.6)$$

In order to analyze the classical limit we have to evaluate (3.6) as $\varepsilon \rightarrow 0$. Consider the first product in (3.6) and let us prove that to lowest order in ε

$$\prod_i P(r_i) = \varepsilon^{N_c} \prod_i G(a_i, m_j) , \quad (3.7)$$

which is the contribution of the εG term in P . We have to show that the $\mathcal{C}_{N_c}(x)$ contribution is of higher order in ε . Suppose a_i is a root of $\mathcal{C}_{N_c}(x)$ of multiplicity n , then it follows from (3.5) that

$$r_i = a_i + c_i \varepsilon^{\frac{1}{2n-1}} + \dots . \quad (3.8)$$

Thus the contribution of $\mathcal{C}_{N_c}(r_i)^2$ is of order $\frac{2n}{2n-1} > 1$ in ε which is higher than the contribution of the εG term. In order to get the lowest order in ε we evaluate the second product in (3.6) at the roots b_i , which yields using (3.4)

$$\prod_j P(b_j) = \Delta[\mathcal{C}_{N_c}]^2 , \quad (3.9)$$

where $\Delta[\mathcal{C}_{N_c}]$, given by (2.26) is the discriminant of the classical curve (2.11). Thus, (3.7) and (3.9) yield

$$\Delta[P] = \varepsilon^{N_c} \Delta[\mathcal{C}_{N_c}]^2 \prod_i G(a_i, m_j) . \quad (3.10)$$

A comparison of (3.10) to (2.25), (2.26) and (2.27) yields

$$G(x, m_i) = \prod_{i=1}^{N_f} (x + m_i) , \quad (3.11)$$

which completes the proof.

The meromorphic one-form λ , satisfying (2.20), takes the form reads

$$\lambda = \frac{xdx}{2\pi iy} \left(\frac{\mathcal{C}_{N_c} G'}{2G} - \mathcal{C}'_{N_c} \right) , \quad (3.12)$$

for $N_f < N_c$. The fact that the residues of λ satisfy (2.21) is a corollary of the residue calculation that will be presented in the next section. This provides a consistency check on our result.

3.2 Examples

$N_c = 2$: Denote the gauge invariant order parameter $\frac{u_2}{2}$ of (2.2) by u .
 $N_f = 0$: The curve is given by [4, 5]

$$y^2 = (x^2 - u)^2 - \Lambda_0^4. \quad (3.13)$$

Let us, as an example, compute the periods and the proportionality constant of (2.19) in this case. We will show in section six that the result is in fact general. The roots of (3.13) are $\pm x_1 = \pm\sqrt{u + \Lambda_0^2}$ and $\pm x_2 = \pm\sqrt{u - \Lambda_0^2}$.

Define the α and β cycles as the contours from $-x_1$ to x_1 and from x_1 to x_2 respectively and back counterclockwise. The corresponding periods of the curve read

$$\begin{aligned} \omega_1 &= 2 \int_{-x_1}^{x_1} \frac{dx}{\sqrt{(x^2 - u)^2 - \Lambda_0^4}} = \frac{2\pi}{x_2} F\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{x_1^2}{x_2^2}\right), \\ \omega_2 &= 2 \int_{x_1}^{x_2} \frac{dx}{\sqrt{(x^2 - u)^2 - \Lambda_0^4}} = -\frac{2\pi i}{x_1 + x_2} F\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{(x_1 - x_2)^2}{(x_1 + x_2)^2}\right), \end{aligned} \quad (3.14)$$

where F is the hyper-geometric function. The behavior of the periods as $u \rightarrow \infty$ is

$$\omega_1 \rightarrow \frac{2}{\sqrt{u}} \log\left(\frac{u}{2\Lambda_0^2}\right), \quad \omega_2 \rightarrow -\frac{\pi i}{\sqrt{u}}. \quad (3.15)$$

Thus,

$$\begin{aligned} a &\simeq -2\pi i \sqrt{u}, \\ a_D &\simeq i \frac{4}{\pi} a \log(a). \end{aligned} \quad (3.16)$$

Since at weak coupling $a = \sqrt{u}$ (2.5) we have

$$\frac{da_D}{du} = -\frac{1}{2\pi i} \oint_{\alpha} \frac{dx}{y}, \quad \frac{da}{du} = -\frac{1}{2\pi i} \oint_{\beta} \frac{dx}{y}. \quad (3.17)$$

$N_f = 1$: As in the proof of (3.2) in order to determine the curve we make use of R -symmetry, the form of instanton corrections and the scaling and classical limits. The curve reads

$$y^2 = (x^2 - u)^2 - \Lambda_1^3(x + m). \quad (3.18)$$

The coefficient of $m\Lambda_1^3$ in (3.18) is determined by integrating the massive quark and requiring compatibility with (3.13), while that of $x\Lambda_1^3$ is fixed by the requirement for

having a singularity at $u = m^2$ at the classical limit $\Lambda_1 \rightarrow 0$ corresponding to a massless quark. The discriminant of the curve (3.18)

$$\Delta_{2,1} = -\Lambda_1^6(256u^3 - 256u^2m^2 - 288um\Lambda_1^3 + 256m^3\Lambda_1^3 + 27\Lambda_1^6) , \quad (3.19)$$

has a novel symmetry: It is invariant under the transformation $u \rightarrow m\Lambda_1, m \rightarrow \frac{u}{\Lambda_1}$. This implies that the corresponding curves are equivalent : There exists an $SL(2, C)$ transformation taking one curve to the other. That means that all the physical quantities associated with the quantum moduli space of vacua such as the BPS spectrum are invariant under this transformation. In particular, the massive theory with mass m at the origin of the moduli space $u = 0$ is equivalent to a massless theory at $u = m\Lambda_1$ on the moduli space. We expect such relations to hold for other theories.

$N_c = 3$: Denote the gauge invariant order parameter $\frac{u_3}{3}$ of (2.2) by v .

$N_f = 0$: The curve takes the form [4, 5]

$$y^2 = (x^3 - ux - v)^2 - \Lambda_0^6 . \quad (3.20)$$

$N_f = 1$: The curve reads

$$y^2 = (x^3 - ux - v)^2 - \Lambda_1^5(x + m) . \quad (3.21)$$

The coefficient of $m\Lambda_1^5$ is determined by integrating a massive quark and requiring compatibility with the $N_f = 0$ case. The coefficient a of $x\Lambda_1^5$ is determined by imposing the expected classical singularity, as we will show now. The singular complex line associated with a classical massless quark is the variety defined by

$$\prod_{i=1}^3 (a_i + m) \equiv v - mu + m^3 = 0 . \quad (3.22)$$

The lowest order term in Λ_1 of the discriminant of the curve (3.21) reads

$$(4u^3 - 27v^2)^2(m^3 - a^2um + a^3v) , \quad (3.23)$$

where we factored out $64\Lambda_1^{15}$. The zero locus of the first term corresponds to singularities where classically the $SU(3)$ group is not spontaneously broken to $U(1)^3$ but rather to the subgroup $SU(2) \times U(1)$, and to the case where it is not broken at all at the origin $u = v = 0$.

The zero locus of the second term is a complex line in the moduli space where a quark becomes massless classically. Comparison with (3.22) fixes the value of the coefficient to

one.

$N_f = 2$: The curve reads

$$y^2 = (x^3 - ux - v)^2 - \Lambda_2^4(x + m_1)(x + m_2) . \quad (3.24)$$

The terms $ax^2 + bu$ which are allowed by R -symmetry and instanton corrections are excluded by imposing the classical singularity structure.

The discriminant of (3.24) for equal masses $m_1 = m_2 = m$ reads

$$\Delta_{3,2} = 64\Lambda_2^{12} (v - mu + m^3)^2 \Delta_+ \Delta_- , \quad (3.25)$$

where

$$\Delta_{\pm} = \left[27(v \pm m\Lambda_2^2)^2 - 4(u \pm \Lambda_2^2)^3 \right] . \quad (3.26)$$

The multiplicity two in the discriminant (3.25) is a reflection of the underlying $SU(2)$ global flavor symmetry: The massless states along this line are in the fundamental representation of this group. As $\Lambda \rightarrow 0$, (3.25) provides an example to the classical singularity structure (2.25).

4 The quantum moduli space for $N_f = N_c$

4.1 The general case

The most general curve consistent with R -symmetry, instanton corrections and classical singularity is

$$y^2 = \mathcal{C}_{N_c}^2(x) - \Lambda^{N_c} \left(\prod_{i=1}^{N_f} (x + m_i) + a\mathcal{C}_{N_c}(x) \right) + b\Lambda^{2N_c} , \quad (4.1)$$

where a and b are constant coefficients. Since the one instanton correction Λ^{N_c} has R charge N_c , the curve (4.1) gets contributions also from a two instanton process of the form Λ^{2N_c} . The structure of the correction $\Lambda^{N_c}\mathcal{C}_{N_c}(x)$ in (4.1) is determined such that it vanishes for $x = a_i$ as required, via the analysis of the previous section, by comparing (3.10) with (2.27).

The curve (4.1) can be written in a form suitable for generalization to $N_f > N_c$:

$$y^2 = F_{N_c}^2 - H_{N_c} , \quad (4.2)$$

where

$$\begin{aligned} F_{N_c} &= \mathcal{C}_{N_c} + \alpha\Lambda^{N_c} \\ H_{N_c} &= \Lambda^{N_c} \left(\prod_{i=1}^{N_f} (x + m_i) + \beta\Lambda^{N_c} \right) , \end{aligned} \quad (4.3)$$

with α and β constants.

The meromorphic one-form λ (2.20) reads in this case

$$\lambda = \frac{xdx}{2\pi iy} \left(\frac{F_{N_c} H'_{N_c}}{2H_{N_c}} - F'_{N_c} \right) . \quad (4.4)$$

The residue formula (2.21) can be used in order to determine the coefficient β , as we will show now. This will be an example of a powerful constraint which will be much used in the sequel.

Let us consider the case of equal bare masses $m_i = m$. The zeros of H_{N_c} in (4.3) read

$$x_i = -m + e^{\frac{2\pi i}{N_f}} \beta^{\frac{1}{N_f}} \Lambda , \quad i = 1, \dots, N_f - 1 . \quad (4.5)$$

The residue of λ in (4.4) at the root x_0 of (4.5) is

$$2\pi i \operatorname{res}_{x=x_0}(\lambda) = \frac{m - \beta}{2} , \quad (4.6)$$

thus (2.21) implies $\beta = 0$. The residues of λ at zeros of y vanish. This is easily seen by differentiating (4.2) with respect to x , which together with (4.2) at $y = 0$ yield

$$\left(\frac{F_{N_c} H'_{N_c}}{2H_{N_c}} - F'_{N_c} \right)_{y=0} = 0 . \quad (4.7)$$

Thus, we verified that the residue formula (2.21) is satisfied completely by λ of (4.4). Note that we are still left with one undetermined constant coefficient α . When $N_c = 2$ it is fixed by imposing a Z_2 symmetry on the singular locus of the curve, as will be shown in section 4.2.

Compatibility with the hyper-elliptic curves for $N_c > N_f$ which will be discussed in the next section implies that $\alpha = \frac{1}{4}$ for $N_c > 2$, however, we do not have a full proof for that.

Thus, we suggest that the family of curves describing the quantum moduli space of vacua for the $N_c = N_f$, $N_c > 2$ is

$$y^2 = \left(\mathcal{C}_{N_c}(x) + \frac{\Lambda^{N_c}}{4} \right)^2 - \Lambda^{N_c} \prod_{i=1}^{N_f} (x + m_i) . \quad (4.8)$$

4.2 Examples

$N_c = N_f = 2$: The curve takes the form

$$y^2 = \left(x^2 - u + \frac{\Lambda_2^2}{8} \right)^2 - \Lambda_2^4 (x + m_1)(x + m_2) . \quad (4.9)$$

R -symmetry, instanton corrections, classical singularity and scaling limit, leave us with a two parameter family of curves given by

$$y^2 = (x^2 - u + a\Lambda_2^2)^2 - \Lambda_2^2(x + m_1)(x + m_2) - b\Lambda_2^4 . \quad (4.10)$$

The singularity structure of the massless theory: Two singular points of multiplicity two, leads to $b = 0$. The requirement that the singular points be located in a Z_2 symmetric form in the moduli space leads to $a = \frac{1}{8}$. There are two cases where an underlying global $SU(2)$ structure appears. First, when $m_2 = m_1 = m$ the discriminant of the curve takes the form

$$\frac{\Lambda_2^4}{16}(8u - 8m\Lambda_2 + \Lambda_2^2)(8u + 8m\Lambda_2 + \Lambda_2^2)(8u - 8m^2 - \Lambda_2^2)^2 . \quad (4.11)$$

The multiplicity two reflects the fact that the massless states along the corresponding singular line are in the fundamental representation of the flavor symmetry group. Second, when $-m_2 = m_1 = m$ the discriminant takes the form

$$\frac{\Lambda_2^4}{16}(64u^2 - 16u\Lambda_2^2 + 64m^2\Lambda_2^2 + \Lambda_2^4)(8u - 8m^2 - \Lambda_2^2)^2 , \quad (4.12)$$

with the multiplicity two showing up.

$N_c = N_f = 3$:

The curve takes the form

$$y^2 = \left(x^3 - ux - v + \frac{\Lambda_3^3}{4}\right)^2 - \Lambda_3^3 \prod_{i=1}^3 (x + m_i) . \quad (4.13)$$

Following the discussion in section 4.1, we are left with the coefficient of Λ_3^3 as the only undetermined parameter. It is determined to be $\frac{1}{4}$ by matching to the $N_c = 3, N_f = 4$ curve upon integration of a massive quark. The latter curve will be determined in the subsequent section.

5 The quantum moduli space for $N_f > N_c$

When the number of flavors is increased the curves describing the quantum moduli spaces may get contributions from higher multi-instanton processes. This increases the number of terms in the polynomials describing the curves that should be determined. In this section we study the general $N_f > N_c$ cases. We parameterize the curves and determine their structure up to a certain number of unknown constant coefficients. Global symmetry considerations suggest a fixed value for these constants, but we do not have a solid proof for that.

5.1 $N_f = N_c + 1$

As a preliminary to the general case, let us add another flavor to the $N_f = N_c$ case which has been analyzed in the previous section. Now the curve is parameterized by two unknown constants. The curve has the form

$$\begin{aligned} y^2 &= F_{N_c}^2 - H_{N_f}, \quad F_{N_c} = \mathcal{C}_{N_c} + \Lambda^{N_c-1}(ax + bt_1(m)) \\ H_{N_f} &= \Lambda^{N_c-1} \prod_{i=1}^{N_f} (x + m_i). \end{aligned} \quad (5.1)$$

Where t_k denotes the symmetric function in m_i of order k

$$t_k(m) = \sum_{i_1 < \dots < i_k} m_{i_1} \cdots m_{i_k}. \quad (5.2)$$

The coefficient b is determined to be $\frac{1}{4}$ by matching to the $N_f = N_c$ case upon integrating a massive quark. The coefficient a takes the value $\frac{1}{4}$ following the discussion in section 2.3: We set the masses of all the quarks but one to zero and consider the complex line $s_i = 0, i \neq N_c$. We expect the discriminant to have multiplicities (N_c, N_c) which is indeed the case only for this value of a .

The final form of the curve is

$$y^2 = \left[\mathcal{C}_{N_c} + \Lambda^{N_c-1} \left(\frac{x}{4} + \frac{t_1(m)}{4} \right) \right]^2 - \Lambda^{N_c-1} \prod_{i=1}^{N_f} (x + m_i). \quad (5.3)$$

5.2 Examples

$N_c = 2, N_f = 3$: R -symmetry, instanton corrections and classical singularity structure restrict the form of the curves to

$$\begin{aligned} y^2 &= \left(x^2 - u + \Lambda_3 \left(\frac{(m_1 + m_2 + m_3)}{8} + bx \right) \right)^2 - \Lambda_3 [(x + m_1)(x + m_2)(x + m_3) \\ &+ a\Lambda_3^3 + b\Lambda_3^2(m_1 + m_2 + m_3) + c\Lambda_3(m_1m_2 + m_1m_3 + m_2m_3) + d\Lambda_3u \\ &+ (e\Lambda_3^2 + f\Lambda_3(m_1 + m_2 + m_3))x + g\Lambda_3x^2]. \end{aligned} \quad (5.4)$$

Applying the above residue considerations we find that most of the terms in (5.4) vanish, and imposing a singularity of order four in the massless case, which follows from the underlying $SO(6)$ symmetry [3], leads to $b = 0$ or $b = \frac{1}{4}$. The $SO(4)$ global symmetry for the flavor masses $(m, 0, 0)$ implies a multiplicity structure $(2, 2)$ in the discriminant and leads to $b = \frac{1}{4}$. Thus, the final curves takes the form

$$y^2 = \left(x^2 - u + \Lambda_3 \left(\frac{(m_1 + m_2 + m_3)}{8} + \frac{x}{4} \right) \right)^2 - \Lambda_3(x + m_1)(x + m_2)(x + m_3). \quad (5.5)$$

$N_c = 3, N_f = 4$:

The curve takes the form

$$y^2 = \left[x^3 - ux - v + \frac{\Lambda_4^2}{4} (x + t_1(m)) \right]^2 - \Lambda_4^2 \prod_{i=1}^4 (x + m_i) . \quad (5.6)$$

Its derivation is a special case of the discussion in section 5.1.

5.3 The general case $N_f > N_c$

The general structure of the family of curves describing the quantum moduli space of vacua when $N_f > N_c$ is encoded in the following claim.

Claim: The curve describing the quantum moduli space with gauge group $SU(N_c)$ and $N_f > N_c$ flavors is:

$$y^2 = \left(\mathcal{C}_{N_c}(x) + \Lambda^{2N_c-N_f} P \right)^2 - \Lambda^{2N_c-N_f} \prod_{i=1}^{N_f} (x + m_i) \quad (5.7)$$

where $P(x, m_i, \Lambda)$ is a polynomial of degree $N_f - N_c$ in x, m_i and is *independent* of the moduli s_k .

Proof: Consider the most general hyper-elliptic curve. Using $\partial_k \partial_l \lambda = \partial_l \partial_k \lambda$ where $\partial_k \equiv \frac{\partial}{\partial s_k}$ together with (2.20) yields

$$x^{-k} \partial_l y^2 = x^{-l} \partial_k y^2 . \quad (5.8)$$

Equation (5.8) implies that y^2 depends on the moduli s_k only via $\mathcal{C}_{N_c}(x)$. Thus, $y^2(s_k) = y^2(\mathcal{C}_{N_c}(x))$. Moreover, R charge symmetry implies that only terms up to quadratic in $\mathcal{C}_{N_c}(x)$ can appear:

$$y^2 = \mathcal{C}_{N_c}^2 g_0 + \mathcal{C}_{N_c} g_1(x, m_i, \Lambda) + g_2(x, m_i, \Lambda) , \quad (5.9)$$

where g_0 is a polynomial of degree 0 in x , namely a constant *, g_1, g_2 are polynomials in x of degree N_c and $2N_c$, respectively and are independent of the moduli s_k . The classical limit fixes $g_0 = 1$. The curve (5.9) may be recast in the form

$$y^2 = \left(\mathcal{C}_{N_c} + \Lambda^{2N_c-N_f} P \right)^2 - \Lambda^{2N_c-N_f} G , \quad (5.10)$$

with

$$G(x, m_i, \Lambda) = \prod_{i=1}^{N_f} (x + m_i) + \sum_{k=1}^{\left[\frac{2N_c}{2N_c-N_f} \right] - 1} \Lambda^{k(2N_c-N_f)} n_k(x, m) , \quad (5.11)$$

*As we will discuss in the sequel, when $N_f = 2N_c$, g_0 may be a modular form $g_0(q)$.

where we used R charge considerations and the form of instanton corrections. The first term in (5.11) has been deduced from the structure of the classical singularity in section 3. The other terms are arranged according to the order of multi-instanton contribution. n_k is a polynomial of degree $(k+1)N_f - 2kN_c$ and $\left\lfloor \frac{2N_c}{2N_c - N_f} \right\rfloor$ denotes its integer value. $P(x, m_i, \Lambda)$ is a polynomial of degree N_f in x, m_i .

We will now use the residue formula (2.21) in order to determine the form of $G(x)$ in (5.10). The meromorphic one-form λ (2.20) associated with (5.10) is

$$\lambda = \frac{xdx}{2\pi iy} \left(\frac{FG'}{2G} - F' \right), \quad (5.12)$$

where, as before, prime denotes derivative with respect to x and $F = \mathcal{C}_{N_c} + \Lambda^{2N_c - N_f} P$.

Consider the zeros x_i of $G(x)$ where $y(x_i) \neq 0$ and let us evaluate $\text{res}(\lambda)$ at these points. Since $F/y = \pm 1$ at x_i and G'/G equal the order n of the root x_i we get

$$2\pi i \text{res}_{x=x_i}(\lambda) = \pm \frac{n}{2} x_i. \quad (5.13)$$

Comparing (5.13) to (2.21) we conclude that the roots of $G(x)$ do not receive quantum corrections, thus $n_k = 0$ in (5.11). This completes the proof of the claim.

In order to fully construct the curve we still have to determine the polynomial P . Studies of the structure of the singularities, compatibility with the global symmetries and the observation on a relation between them, made in section 2.3, suggest the form of P and the curve for $N_f > N_c$ with $N_c > 2$:

$$y^2 = \left[\mathcal{C}_{N_c} + \frac{\Lambda^{2N_c - N_f}}{4} \sum_{i=0}^{N_f - N_c} x^{N_f - N_c - i} t_i(m) \right]^2 - \Lambda^{2N_c - N_f} \prod_{i=1}^{N_f} (x + m_i). \quad (5.14)$$

Note that (5.14) includes (3.2) and (4.8) as special cases, and that it gets only one and two-instanton contributions. However, we do not have a proof for this formula.

5.4 Examples

$N_c = 3, N_f = 5$: The most general curve consistent with the general structure (5.7) reads

$$y^2 = \left[x^3 - ux - v + \Lambda_5(ax^2 + x(bt_1(m) + c\Lambda_5) + dt_2(m) + e\Lambda_5 t_1(m) + f\Lambda_5^2) \right]^2 - \Lambda_5 \prod_{i=1}^5 (x + m_i). \quad (5.15)$$

Let us now imply global symmetry considerations in order to determine the curve. Consider the complex line $u = 0$ and set the flavor masses to the values $(m, m, 0, 0, 0)$. We expect that the discriminant of the curve exhibit multiplicity structure of the form $(3, 3, 2)$. The values of e and f do not affect the analysis since they correspond to shifts in the moduli parameter v , and thus can be set to zero for the meantime. Requiring the above multiplicity structure implies that $a = b = c = \frac{1}{4}$ and $d = 0$. In order to determine the values of e and f consider the complex line $v = 0$ with mass values as before. Requiring the same multiplicity structure yields the values $e = f = 0$. Thus, the final form of the curve is

$$y^2 = \left[x^3 - ux - v + \frac{\Lambda_5}{4}(x^2 + xt_1(m) + t_2(m)) \right]^2 - \Lambda_5 \prod_{i=1}^5 (x + m_i) . \quad (5.16)$$

6 Weak coupling monodromies

As a consistency check of the hyper-elliptic curves describing the quantum moduli spaces of vacua which were constructed in previous sections, we will compute in this section the weak coupling monodromies of the curves and compare to the ones expected on physical grounds.

Let l be a complex line in the moduli space defined by $s_i = 0, i \neq N_c, s_{N_c} = s$. We consider the massless case, however the generalization to the massive theories is straightforward.

In the weak coupling limit $s \rightarrow \infty$ ($s \gg \Lambda^{N_c}$) the hyper-elliptic curve for N_c, N_f theory takes the form

$$y^2 = (x^{N_c} + s)^2 - \Lambda^{2N_c - N_f} x^{N_f} . \quad (6.1)$$

The roots of the curve (6.1), to first order corrections in Λ , are

$$\begin{aligned} x_{1,l} &= \varepsilon^l z_1^{\frac{1}{N_c}} , \\ x_{2,l} &= \varepsilon^l z_2^{\frac{1}{N_c}} \quad l = 1, \dots, N_c , \end{aligned} \quad (6.2)$$

where $\varepsilon = e^{\frac{2\pi i}{N_c}}$ and

$$\begin{aligned} z_1 &= -s \left(1 + \Lambda^{\frac{2N_c - N_f}{2}} (-s)^{\frac{N_f - 2N_c}{2N_c}} \right) , \\ z_2 &= -s \left(1 - \Lambda^{\frac{2N_c - N_f}{2}} (-s)^{\frac{N_f - 2N_c}{2N_c}} \right) . \end{aligned} \quad (6.3)$$

Define the α_l and β_l cycles as the paths from $x_{1,l}$ to $x_{2,l}$ and from $x_{1,1}$ to $x_{1,l}$ and back counterclockwise, respectively, such that the intersection form of the homology cycles

reads $(\beta_k, \alpha_l) = \delta_{kl}$ with $k, l = 2, \dots, N_c$. Evidently, the α cycles are linearly dependent and satisfy $\sum_{l=1}^{N_c} \alpha_l = 0$.

One way to compute the weak coupling monodromy is to trace the cycles as $s \rightarrow e^{2\pi i} s$ around infinity, as was done in [5]. We will take another route: We will compute the periods of the curve and study their transformation properties as s traverses a cycle around infinity. Denote $A_{kl} = \int_{\alpha_l} \frac{x^{N_c-k} dx}{y}$ and $B_{kl} = \int_{\beta_l} \frac{x^{N_c-k} dx}{y}$. At weak coupling *

$$\begin{aligned} B_{kl} &= 2 \int_{x_{1,l}}^{x_{1,l}} \frac{x^{N_c-k} dx}{y} = (\varepsilon^{l(N_c-k+1)} - \varepsilon) \frac{z_1^{m+\frac{1}{2}}}{z_2^{\frac{1}{2}}} B\left(m+1, \frac{1}{2}\right) F\left(\frac{1}{2}, m+1, m+\frac{3}{2}, \frac{z_1}{z_2}\right), \\ A_{kl} &= 2 \int_{x_{1,l}}^{x_{2,l}} \frac{x^{N_c-k} dx}{y} = -\frac{2\pi i}{N_c} \varepsilon^{l(N_c-k+1)} z_1^m F\left(-m, \frac{1}{2}, 1, 1 - \frac{z_2}{z_1}\right), \end{aligned} \quad (6.4)$$

where $B(m, n)$ is the beta function and $m = \frac{1-k}{N_c}$. Setting $k = l = N_c = 2, N_f = 0$ in (6.4) yields (3.14).

Let us consider the $k = N_c$ case and expand (6.4) around infinity in s , then recalling (2.19) †

$$\begin{aligned} a^l &\propto \int A_{N_c l} ds \simeq 2\pi i \varepsilon^l (-s)^{\frac{1}{N_c}}, \\ a_D^l &\propto \int B_{N_c l} ds \simeq N_c (2N_c - N_f) (\varepsilon - \varepsilon^l) (-s)^{\frac{1}{N_c}} \log((-s)^{\frac{1}{N_c}}). \end{aligned} \quad (6.5)$$

First, we can deduce from (6.5) the proportionality constant in (2.19). The classical relation (2.6) imply that $a^l = \varepsilon^l (-s)^{\frac{1}{N_c}}$. Thus, comparison to (6.5) fixes the constant to $\frac{1}{2\pi i}$.

Second, as we shall see now, (6.5) yields the monodromy at weak coupling as expected from the one loop effective action (2.8). Denote $a = (-s)^{\frac{1}{N_c}}$, then (6.5) can be written as

$$\begin{aligned} a^l &= \varepsilon^l a, \\ a_D^l &= \frac{i}{2\pi} N_c (2N_c - N_f) (\varepsilon^l - \varepsilon) a \log(a), \end{aligned} \quad (6.6)$$

which is consistent with (2.10) and (2.16). Thus we conclude that the periods of the hyper-elliptic curves that we constructed have the correct monodromy at infinity, as expected on physical ground. Note that a_D in (6.6) can be written as

$$a_D^l = \frac{i}{2\pi} N_c (2N_c - N_f) (a^l + \sum_{i=2}^{N_c} a^i) \log(a). \quad (6.7)$$

*The integrals (6.4) are evaluated by changing variables to $z = x^{N_c}$ and keeping track of the contours of integration.

†We have rescaled a_D by factor two. This change of notation is needed in order to get the correct monodromies in the sequel.

As $s \rightarrow e^{2\pi i} s$ on a large cycle in s plane

$$\begin{aligned}
a^l &\rightarrow a^{l+1}, \quad l = 2, \dots, N_c - 1 \\
a^{N_c} &\rightarrow -\sum_{i=2}^{N_c} a^i, \\
a_D^l &\rightarrow a_D^{l+1} - a_D^1 - (2N_c - N_f)(a^{l+1} - a^1), \quad l = 2, \dots, N_c - 1 \\
a_D^{N_c} &\rightarrow -a_D^1 + (2N_c - N_f)(a^1 + \sum_{i=2}^{N_c} a^i).
\end{aligned} \tag{6.8}$$

The matrix representation of the monodromy at infinity is read from (6.8). It takes the form

$$M = \begin{pmatrix} (A^{-1})^t & B \\ 0 & A \end{pmatrix}_{2g \times 2g}, \tag{6.9}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & & & 1 \\ -1 & -1 & \cdots & -1 \end{pmatrix}_{g \times g}, \quad B = -(2N_c - N_f) \begin{pmatrix} -1 & 1 & 0 & \cdots \\ -1 & 0 & 1 & \cdots \\ \vdots & & & 1 \\ -2 & -1 & \cdots & -1 \end{pmatrix}_{g \times g}. \tag{6.10}$$

In analogy to the $N_c = 2$ case [3], the monodromy matrix (6.9) can be written as

$$M = PT^{2N_c - N_f}, \tag{6.11}$$

where

$$P = \begin{pmatrix} (A^{-1})^t & 0 \\ 0 & A \end{pmatrix}_{2g \times 2g}, \quad T = \begin{pmatrix} I & C \\ 0 & I \end{pmatrix}_{2g \times 2g} \tag{6.12}$$

and

$$C = \begin{pmatrix} 2 & 1 & 1 & \cdots \\ 1 & 2 & 1 & \cdots \\ \vdots & & 2 & 1 \\ 1 & 1 & \cdots & 2 \end{pmatrix}_{g \times g}. \tag{6.13}$$

P is the "classical" part of the monodromy associated with Weyl transformation of the roots of the classical curve (2.11). T is the "quantum" part of the monodromy associated with the one loop \log correction to the pre-potential (2.10).

Finally, let us make a comment on the strong coupling regime. When $N_f = 0$ or $N_f = N_c$, the roots of (6.1) can be calculated exactly. Thus, following the above calculation, the periods of these curves can be computed exactly. However, preliminary studies of the strong coupling monodromies in these cases indicate that there might be a problem with the computation which is not detected at weak coupling.

7 The quantum moduli space for $N_f = 2N_c$

7.1 The general case

When $N_f = 2N_c$ and the bare masses are zero we get conformally invariant theories. In these cases the classical relations (2.5), (2.6) and (2.9) are expected to be valid quantum mechanically. Thus,

$$a_D^i = \tau_{ij} a^j, \quad (7.1)$$

where

$$\tau_{ij} = \tau (\delta_{ij} + 1), \quad (7.2)$$

is the matrix of theta angles and coupling constants of the theory derived from (2.9), and

$$a_D^j = \tau \left(a^j + \sum_{i=2}^{N_c} a^i \right). \quad (7.3)$$

The classical and quantum moduli spaces are identical and are described by a hyperelliptic curve with period matrix τ_{ij} (7.2) and periods as in (2.19) with a^i, a_D^i satisfying (2.6) and (7.3).

The structure of the family of curves describing the moduli space of vacua for $N_f = 2N_c$ is encoded in the following:

Claim: The curve for the quantum moduli space for $SU(N_c)$ gauge group with $N_f = 2N_c$ is:

$$y^2 = [\mathcal{C}_{N_c}(x, l(q)s_k) + P(x, m, q)]^2 - L(q) \prod_{i=1}^{2N_c} (x + l(q)m_i). \quad (7.4)$$

$l(q)$ is a modular form satisfying $l(q) \rightarrow 1$ as $q \rightarrow 0$. $P(x, m_i, q)$ is modular form satisfying $P \propto q$ as $q \rightarrow 0$ and a polynomial of degree N_c in x which is *independent* of the moduli s_k . $L(q)$ is a modular form of weight zero satisfying $L(q) \propto q + O(q^2)$.

Proof: The proof is similar to that of section 5.3. The main difference is that the dynamically generated scale Λ is replaced by q as defined in (2.32). The structure of the first term in (7.4) is deduced following the same argument as in equations (5.8) - (5.10) together with R -symmetry. The factor g_0 of (5.9) gets contributions both from $\mathcal{C}_{N_c}(x, l(q)s_k)$ and $P(x, m, q)$. The modification of all the moduli s_k by the same modular form $l(q)$ is consistent with R -symmetry as well as (5.8) and (5.9).

The structure of the second term follows from the analysis of the residues (5.13) of the meromorphic one-form λ (5.12), which now gets a pre-factor $\frac{1}{l(q)}$. This implies that m_i must be rescaled by $l(q)$ in order to ensure that the residues of λ be independent of τ . The

behavior of $L(q)$ as $q \rightarrow 0$ is implied by the matching condition (2.32) when integrating a massive quark.

The compatibility with (5.14) upon integrating massive quarks and that of the singularity structure with the global symmetry suggest that the curve for $N_f = 2N_c$ takes the form

$$y^2 = \left[\mathcal{C}_{N_c}(x, l(q)s_k) + \frac{L(q)}{4} \sum_{i=0}^{N_c} x^{N_c-i} t_i(m) \right]^2 - L(q) \prod_{i=1}^{2N_c} (x + l(q)m_i) . \quad (7.5)$$

As for the $N_c < N_f < 2N_c$ cases we do not have a proof of (7.5). The form of $L(q)$ and coefficient $l(q)$ of the moduli s_k should be determined such that the period matrix of the massless curve takes the form (7.1) and the periods satisfy (2.6) and (7.2).

Consider now the curve (7.5) with the masses being set to zero. Since equations (2.6), (7.1) and (7.2) should hold at any generic (non singular) point in the moduli space let us, for simplicity, use the complex line l defined previously by $s_i = 0, i \neq N_c, s_{N_c} = s$.

When the masses are set to zero the curve (7.5) reads

$$y^2 = \left[\left(1 + \frac{L}{4}\right) x^{N_c} + ls \right]^2 - Lx^{2N_c} . \quad (7.6)$$

Its roots take the form

$$\begin{aligned} x_{1,l} &= \varepsilon^l z_1^{\frac{1}{N_c}} , \\ x_{2,l} &= \varepsilon^l z_2^{\frac{1}{N_c}} \quad l = 1, \dots, N_c , \end{aligned} \quad (7.7)$$

where $\varepsilon = e^{\frac{2\pi i}{N_c}}$ and

$$z_1 = -\frac{l(q)s}{\left(1 + \frac{\sqrt{L}}{2}\right)^2} \equiv -\tilde{z}_1 s , \quad z_2 = -\frac{l(q)s}{\left(1 - \frac{\sqrt{L}}{2}\right)^2} \equiv -\tilde{z}_2 s . \quad (7.8)$$

Evaluating a^l, a_D^l as in section six we get

$$\begin{aligned} a^l &= f(q) \varepsilon^l (-s)^{\frac{1}{N_c}} , \\ a_D^l &= g(q) (\varepsilon^l - \varepsilon) (-s)^{\frac{1}{N_c}} , \end{aligned} \quad (7.9)$$

where f and g are read from (6.4). Note that in contrast to the $N_f < 2N_c$ theories, a_D^l do not get logarithmic corrections. In order that the classical relation will hold quantum mechanically we have to require that $f(q) \equiv 1$, while in order that (7.1) be correct we need $\frac{g(q)}{f(q)} = \tau$.

Let us now suggest possible forms for $L(q)$ and $l(q)$ which we will verify for the $N_c = 2$ case, but for which we do not have a proof for general N_c . Introduce the theta constant

[11]

$$\theta[m_1 \ m_2] = \sum_{n \in \mathbb{Z}^g} \exp \left\{ 2\pi i \left[\frac{1}{2}(n + m_1)^t \tau (n + m_1) + (n + m_1)^t m_2 \right] \right\} , \quad (7.10)$$

where τ is the period matrix (7.2) and m_1, m_2 are dimension g vectors with zeros and halves as entries. We suggest that *

$$L(q) = \frac{4\theta \left[\frac{1}{2} \ 0 \right]^4}{\theta[0 \ 0]^4} , \quad l(q) = \frac{\theta \left[0 \ \frac{1}{2} \right]^8}{\theta[0 \ 0]^4} , \quad (7.11)$$

where 0 denotes the zero vector and $\frac{1}{2}$ stands for a vector with one of its entries being $\frac{1}{2}$ and the others are zeros. The definition (7.10) with the period matrix (7.2) imply that $\theta \left[\frac{1}{2} \ 0 \right]$ and $\theta \left[0 \ \frac{1}{2} \right]$ are independent of which entry is the $\frac{1}{2}$. $\theta[0 \ 0]^4$, $\theta \left[\frac{1}{2} \ 0 \right]^4$ and $\theta \left[0 \ \frac{1}{2} \right]^4$ are modular forms of $\Gamma_{2,4}$ † with weight two.

For $N_c = 2$ (7.11) reduces to

$$L(q) = \frac{4\theta_{10}^4}{\theta_{00}^4} , \quad l(q) = \frac{\theta_{01}^8}{\theta_{00}^4} , \quad (7.12)$$

with the theta null values [11]‡

$$\begin{aligned} \theta_{00}(q) &= \sum_{n \in \mathbb{Z}} q^{n^2} , \\ \theta_{01}(q) &= \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2} , \\ \theta_{10}(q) &= \sum_{n \in \mathbb{Z}} q^{\left(n + \frac{1}{2}\right)^2} . \end{aligned} \quad (7.13)$$

θ_{00}^4 , θ_{01}^4 and θ_{10}^4 are modular forms of Γ_4 § with weight two and satisfy the Jacobi identity

$$\theta_{00}^4 = \theta_{01}^4 + \theta_{10}^4 . \quad (7.14)$$

We verified that for $N_c = 2$ the requirements from the functions $f(q)$ and $g(q)$ are satisfied with L, l of (7.12). We leave it as an open problem to verify that the requirements are satisfied for general N_c with L, l of (7.11).

*This suggestion is based on the Thomae formula [12] for the construction of a hyper-elliptic curve with a prescribed period matrix, but is not a direct consequence of it.

† $\Gamma_{2,4} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{Z}) \mid A, D = 1_{g \times g} \pmod{2}, B, C = 0 \pmod{2}, 4 \text{ divides the diagonals of } B, C \right\}$.

‡In this case the notation amounts to replacing $\frac{1}{2}$ by 1. Another notation in the literature is: $\theta_2(0, q) \equiv \theta_{10}(q)$, $\theta_3(0, q) \equiv \theta_{00}(q)$ and $\theta_4(0, q) \equiv \theta_{01}(q)$.

§ $\Gamma_4 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid a, d = 1 \pmod{4}, b, c = 0 \pmod{4} \right\}$.

7.2 An example

$N_c = 2$: Consider the case $N_f = 4$. The curve (7.4) takes the form

$$y^2 = \left[a(q)x^2 - b(q)u + c(q) \sum_{i<j} m_i m_j + d(q)x \sum_{i=1}^4 m_i \right]^2 - L(q) \prod_{i=1}^4 (x + b(q)m_i) . \quad (7.15)$$

with the coefficients being modular forms.

Matching to the $N_f = 3$ curve when integrating a massive quark yields

$$\begin{aligned} a(q) &\rightarrow 1 , & b(q) &\rightarrow 1 , & c(q) &\rightarrow \frac{q}{8} , \\ d(q) &\rightarrow \frac{q}{4} , & L(q) &\rightarrow q , \end{aligned} \quad (7.16)$$

as $q \rightarrow 0$.

The global symmetries together with (7.16) yield

$$a(q) = 1 + \frac{L}{4} , \quad c(q) = \frac{L}{8} , \quad d(q) = \frac{L}{4} . \quad (7.17)$$

$L(q)$ and $b(q)$ are determined by requiring that in the massless case the periods of the curve are such that the classical relation $a = \sqrt{u}$ and (7.1) are satisfied. These yield $L(q)$ as in (7.12) and

$$b(q) = \frac{\theta_{01}^8}{\theta_{00}^4} = l(q) . \quad (7.18)$$

Thus, the $N_c = 2, N_f = 4$ curve reads

$$y^2 = \left[\left(1 + \frac{L(q)}{4} \right) x^2 - l(q)u + \frac{L(q)}{8} \sum_{i<j} m_i m_j + \frac{L(q)}{4} x \sum_{i=1}^4 m_i \right]^2 - L(q) \prod_{i=1}^4 (x + l(q)m_i) , \quad (7.19)$$

with $L(q)$ and $l(q)$ given by (7.12).

8 Discussion and conclusions

In this paper we constructed the hyper-elliptic curves which describe the quantum moduli spaces of vacua of $N = 2$ supersymmetric $SU(N_c)$ gauge theories with N_f flavors of quarks in the fundamental representation. We showed that the curves for $N_f < N_c$ are completely determined by R -symmetry, the form of instanton corrections and the requirement for the correct classical singularity structure. The compatibility of the residue calculations with the BPS formula as well as the correct weak coupling monodromy provide further support to the results.

As expected, the complete specification of the curves for $N_f \geq N_c$ is more complicated. As in the $SU(2)$ case [3], the residues of the meromorphic one-form λ provide strong constraints on the structure of the curves. Together with the other principles, discussed in section 2.4, we worked out the structure of the curves, up to certain unknown constant coefficients.

We have not fully exploited the relation between the global symmetries and the singularity structure. However, an observation that we made in section 2.3 on the form of the discriminants suggested a complete determination of all the unknown coefficients. It will be interesting to establish this observation on a firm basis, and to fully understand the physics underlying it.

Weak coupling monodromies were computed for all the curves and were shown to coincide with what is expected on physical grounds, thus providing a check on the results. We left the calculation of the strong coupling monodromies for the future. This will clearly be needed in order to extract the physics of these theories. Along the way we derived the exact metrics on the quantum moduli spaces as well as the exact spectrum of stable massive states.

We found an example of a novel symmetry of a quantum moduli space: Invariance under the exchange of a moduli parameter and the bare mass. This implies a sort of duality that relates theories with the same gauge group and different vacua, to be contrasted with the duality of [13] that relates theories with different gauge groups. Physically such a symmetry is surprising since it relates bare parameters of the classical Lagrangian combined with a dynamically generated mass scale to the vacuum expectation value of the scalar fields. We expect more symmetries of this type to appear in these theories and it will be important to find their general form.

An open question is to find the singular points that correspond to $N = 1$ vacua. As discussed in [6] for the $N_f = 0$ theories, these points are associated with curves of degree $2N_c$ with $N_c - 1$ pairs of identical branch points. A preliminary check of the massless $N_c = 3$ curves with $0 < N_f \leq 6$, that have been constructed in this paper, did not reveal such points. Evidently, we expect to see these points in the massive cases. Identifying these points will clearly shed more light on the physics of these theories in the strong coupling regime. Another direction is to study the theories at the singular points that correspond to non mutually local massless fields [14].

In the appendix we applied our method for constructing the quantum moduli spaces, to $N = 1$ supersymmetric $SU(N_c)$ gauge theories with a single matter field in the adjoint representation of the gauge group and N_f matter fields in the fundamental representation.

This generalizes the constructions of the quantum moduli spaces of $N = 2$ to general mass matrix and Yukawa couplings. It is clear that the method that we use for constructing the curves is rather general and can be applied in a straightforward manner to a vast number of $N = 1$ theories in the coulomb phase with different matter content. Another, less straightforward, direction for generalization is to include other gauge invariant moduli such as meson operators and to construct the quantum moduli spaces for the theories discussed in [15–17].

The ability to extract exact results in the theories that were studied in this paper points to an underlying integrable structure [18]. In particular, one expects that the pre-potential will be related to a τ function of some integrable hierarchy and that the variety describing the quantum moduli space of vacua will arise as a solution to a non-linear integrable equation of the hierarchy. Revealing these structures may provide us with powerful computational tools for these four-dimensional models.

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Appendices

A On $N = 1$ quantum moduli spaces of vacua

The procedure applied in this paper for constructing the curves, which describe the quantum moduli spaces of $N = 2$ supersymmetric gauge theories is rather general. It can be used, for instance, in order to construct the curves which describe the effective Abelian gauge field couplings in the Coulomb phase of various $N = 1$ theories with different matter content.

As an example, we will construct in this appendix the hyper-elliptic curves which describe the quantum moduli spaces of $N = 1$ supersymmetric $SU(N_c)$ gauge theories with one flavor in the adjoint representation of the gauge group and N_f flavors in the fundamental representation, and with general mass matrix m_{ij} and Yukawa couplings λ_{ij} ^{*}. In these cases the super-potential takes the form

$$W = \lambda_{ij} \tilde{Q}^i \Phi Q^j + m_{ij} \tilde{Q}^i Q^j , \quad (\text{A.1})$$

with i being the flavor index and color indices suppressed. Consider first the $N_f < N_c$ theories. As we have seen in section three, R -symmetry, instanton corrections and the classical singularity structure determine the quantum moduli space curve completely. The super-potential (A.1) implies that there is classically a massless state whenever

$$\Delta_{N_f, N_c} = \det_{\alpha\beta} \det_{ij} (\lambda_{ij} \phi_{\alpha\beta} + m_{ij} \delta_{\alpha\beta}) , \quad (\text{A.2})$$

vanishes. Thus, the classical discriminant should have the form (A.2). The function $G(x, m_i)$ of (3.11) generalizes and for $N_c > 2$ takes the form of the characteristic polynomial of the matrix $\lambda^{-1}M$

$$G(x, m, \lambda) = \det(\lambda x + m) = \det \lambda \det(x + \lambda^{-1}m) = \det \lambda \sum_{i=1}^{N_f} t_i x^{N_f-i} . \quad (\text{A.3})$$

where t_k are the symmetric functions of the matrix $\lambda^{-1}m$ defined by the Newton formula (2.7) with u_k being $\text{Tr}(\lambda^{-1}m)^k$. In analogy to (3.2), the curves describing the quantum moduli spaces for $N_f < N_c$ where $N_c > 2$ take the form

$$y^2 = \det_{\alpha\beta} (x - \phi)^2 - \Lambda^{2N_c - N_f} \det_{ij} \lambda \det_{ij} (x + \lambda^{-1}M) . \quad (\text{A.4})$$

^{*}Setting $m_{ij} = \text{diag}[m_1, \dots, m_{N_f}]$ and $\lambda_{ij} = \delta_{ij}$ for $N_c \neq 2$ yields the $N = 2$ theories.

The $N_c = 2$ case is more subtle, as a consequence of the fact that the fundamental representation is pseudo-real. The super-potential in this case reads

$$W = \lambda_{ij} \tilde{Q}^i \Phi Q^j + m_{ij} \tilde{Q}^i Q^j , \quad (\text{A.5})$$

where λ and m are $2N_f \times 2N_f$ symmetric and antisymmetric matrices, respectively. The classical discriminant is given by

$$\Delta_{N_f,2} = \det \lambda \sum_{i=1}^{N_f} t_{2i} (\lambda^{-1} M) (-u)^{N_f-i} , \quad (\text{A.6})$$

and the curve describing the quantum moduli space of $N_c = 2, N_f = 1$ is

$$y^2 = (x^2 - u)^2 - \Lambda^3 \text{Pf}(\lambda) (x + \text{Pf}(\lambda^{-1} m)) . \quad (\text{A.7})$$

The higher flavor $N_c = 2$ curves can be constructed in a complete analogy to the $N = 2$ models. For instance, the curve for $N_c = 2, N_f = 2$ is given by

$$y^2 = \left(x^2 - u + \frac{\Lambda^2 \text{Pf} \lambda}{8} \right)^2 - \Lambda^2 \text{Pf} \lambda \left(x^2 + x \sqrt{2 \text{Pf}(\lambda^{-1} m) - t_2} + \text{Pf}(\lambda^{-1} m) \right) . \quad (\text{A.8})$$

Following (5.14), we suggest that the curves for $N_f \geq N_c$ with $N_c > 2$ take the form

$$y^2 = \left[\det_{\alpha\beta} (x - \phi) + \frac{\Lambda^{2N_c-N_f} \det_{ij} \lambda}{4} \sum_{i=0}^{N_f-N_c} t_i x^{N_f-N_c-i} \right]^2 - \Lambda^{2N_c-N_f} \det_{ij} \lambda \det_{ij} (x + \lambda^{-1} M) . \quad (\text{A.9})$$

The results presented in this appendix generalize those of [7] to $N_c > 2$.

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